Journal of Statistical Physics, Vol. 49, Nos. 3/4, 1987

Stationary Nonequilibrium States in the Ising Model with Locally Competing Temperatures

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Received February 11, 1987

We study kinetic one- and two-dimensional Ising models whose transition probabilities occur according to two (or more) locally competing temperatures. The model is solved analytically and studied numerically on different assumptions to reveal a variety of stationary nonequilibrium states and phase transitions; we also investigate the system relaxation in some typical cases.

KEY WORDS: Stationary nonequilibrium states; Ising model; competing temperatures; nonequilibrium phase transitions.

1. INTRODUCTION AND MODEL SYSTEM

Relatively simple nonequilibrium states (see, e.g., Ref. 1) occur when the system of interest is coupled to some external agent (e.g., two baths at different temperatures), the coupling producing stationary states characterized by a small set of macroscopic parameters. Those situations cannot be dealt with in general with the powerful ensemble theory for equilibrium Gibbs states, but most usually one needs to develop specific approximate methods for each particular problem. Thus, some recent progress in this field concentrates on mathematically well-defined lattice model systems^(2-5) with simple analytical solutions and nontrivial physical behavior.

This paper is also concerned with the description of nonequilibrium stationary states and phase transitions occurring in a simple model system. The model here consists, as in the familiar Ising model,^(6-8) of a simple cubic lattice in *d* dimensions, at each site of which there is a spin variable

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0022-4715/87/1100-0551$05.00/0

$s(x) = \pm 1$, $x \in \mathbf{Z}^d$. Each configuration $\mathbf{s} = \{s(x), x \in \mathbf{Z}^d\}$ has an interaction energy defined by

$$H(\mathbf{s}) = -J \sum_{x,y}{}^{*} s(x)\, s(y) - h \sum_{x} s(x) \tag{1.1}$$

where the starred sum is over nearest-neighbor lattice points x and y, J is the coupling constant, and h represents the external magnetic field. The configurational density probability function $P(\mathbf{s})$ evolves with time according to a Markovian master equation,

$$\frac{dP(\mathbf{s}, t)}{dt} = \sum_{\mathbf{s}'} [w(\mathbf{s} \mid \mathbf{s}')\, P(\mathbf{s}', t) - w(\mathbf{s}' \mid \mathbf{s})\, P(\mathbf{s}, t)] \tag{1.2}$$

where $w(\mathbf{s}' \mid \mathbf{s})$ is the transition probability per unit time for $\mathbf{s} \to \mathbf{s}'$. It is further assumed(7,8) that each transition between consecutive configurations $\mathbf{s} \to \mathbf{s}'$, just inverts the spin at a site, say $s(x) \to -s(x)$, with probability $w(\mathbf{s}' \mid \mathbf{s}) \equiv w(x; \mathbf{s})$ depending on the change of the configurational energy $\Delta H = H(\mathbf{s}') - H(\mathbf{s})$ that would be caused by the inversion. An explicit expression for $w(x; \mathbf{s})$ in Eq. (1.2) leads to equations for the relevant quantities; for instance, one immediately has for the system magnetization

$$\frac{d\langle s(x) \rangle}{dt} = -2\langle s(x)\, \omega(x; \mathbf{s}) \rangle \tag{1.3}$$

where

$$\langle A \rangle \equiv \sum_{\mathbf{s}} A(\mathbf{s})\, P(\mathbf{s}, t) \tag{1.4}$$

The equilibrium properties may then be obtained within the present formalism by requiring that $d\langle A \rangle / dt = 0$, i.e., $\langle s(x)\, w \rangle = 0$, etc.

We shall, however, introduce a nonequilibrium condition in the model by assuming that $w(x; \mathbf{s})$ results from the competition between n different microscopic mechanisms of evolution, namely

$$w(x; \mathbf{s}) = \sum_{i=1}^{n} p_i w^{(i)}(x; \mathbf{s}) \tag{1.5}$$

and

$$\sum_{i=1}^{n} p_i = 1, \qquad P_i \geqslant 0 \quad \text{for all } i \tag{1.6}$$

Each mechanism i is characterized by a canonical transition probability $w^{(i)}$

satisfying individually a detailed balance condition at temperature T_i. Thus, there is a stationary regime for which

$$\sum_{i=1}^{n} p_i \langle s(x)\, w^{(i)}(x; \mathbf{s}) \rangle_{\mathrm{st}} = 0 \tag{1.7}$$

and similar equations for higher products of spins. One notices here that the case where each transition probability would drive the system to the *same* equilibrium state satisfies $\langle s(x)\, w^{(i)}(x; \mathbf{s}) \rangle = 0$ for all i, so that Eq. (1.7) holds trivially, while the stationary nonequilibrium regime may depend on the $w^{(i)}$ and on the p_i otherwise.

Even though the above model may be worked out, at least formally and sometimes explicitly, for different temperature distributions T_i, we shall refer here for the sake of simplicity to the case of only two competing temperatures T_1 and T_2; that is,

$$w(x; \mathbf{s}) = p w^{(1)}(x; \mathbf{s}) + (1 - p)\, w^{(2)}(x; \mathbf{s}) \tag{1.8}$$

This may be interpreted by assuming that the spin flip at each site x is attempted with probability p as if it were in contact with a thermal bath at temperature $T_1 = T - \Delta T$ and with probability $1 - p$ as if the temperature of the bath inducing the transition were $T_2 = T + \Delta T$; $T \geqslant \Delta T > 0$. The limit $\Delta T \to 0$ then recovers the usual equilibrium case, while $\Delta T \neq 0$ reveals a rich nonequilibrium behavior, which will be described in the following sections. The more general case (1.5) with an arbitrary temperature distribution T_i is of course even more interesting, but we limit ourselves to illustrating the simplest case. This will be done, when $h = 0$ and $J > 0$, by solving the model in the case $d = 1$, by considering two different mean field approximate solutions and a perturbative treatment around the equilibrium (Onsager) solution for $d = 2$, and by performing a Monte Carlo simulation in the case $d = 2$; we consider both the case where $\Delta T = \mathrm{const}$ and the case of a temperature-dependent increment, such as $T_{1,2} = (1 \mp \delta)\, T$, $\delta = \mathrm{const}$.

2. SOLUTION FOR $d = 1$

Let us first consider explicitly the one-dimensional case $d = 1$ $(h = 0)$ and Glauber's transition probabilities,$^{(7)}$ defined as

$$w^{(i)} = \tfrac{1}{2}\alpha\{1 - \tfrac{1}{2}\gamma_i s(x)[s(x+1) + s(x-1)]\} \tag{2.1}$$

where

$$\gamma_i = \tanh(2J/kT_i), \qquad J > 0 \tag{2.2}$$

It readily follows from Eq. (1.7) that the stationary regime is characterized by

$$m[1-p\gamma_1-(1-p)\gamma_2]=0 \tag{2.3}$$

where $m\equiv\langle s(x)\rangle_{\text{st}}$, independent of x. The only solution of Eq. (2.3) with physical relevance in the present case ($\Delta T>0$) is $m=0$, i.e., any stationary translation-invariant state of the ferromagnetic model must have zero magnetization for all possible temperatures $0\leqslant T_1<T_2$; even more,[2], since this is a translation-invariant attractive spin system, the uniqueness of that stationary state also follows.(9)

Concerning (higher order) spin correlation functions, one only needs to notice that the global transition probability (1.5) is given by

$$w(x;\mathrm{s})=\tfrac{1}{2}\alpha\{1-\tfrac{1}{2}\gamma_{\text{eff}}s(x)[s(x+1)+s(x-1)]\},\qquad \gamma_{\text{eff}}\equiv\sum_i p_i\gamma_i \tag{2.4}$$

for the choice (2.1); the same solution thus follows as for the equilibrium ($d=1$) model(7) with an effective temperature given by

$$\tanh(2J/kT_{\text{eff}})=\sum_i p_i\tanh(2J/kT_i) \tag{2.5}$$

Since we are dealing with nonequilibrium states, they may depend essentially, however, on the particular choice for $w^{(i)}$. For instance, when one replaces (2.1) by the so-called Metropolis rates, namely

$$w^{(i)}=\tfrac{1}{4}(3+\gamma_i')+\tfrac{1}{4}(\gamma_i'-1)\{s(x)[s(x+1)+s(x-1)]+s(x+1)\,s(x-1)\} \tag{2.6}$$

for $d=1$ where

$$\gamma_i'\equiv\exp(-4J/kT_i) \tag{2.7}$$

the same solution follows again as for the equilibrium case, except that the effective temperature is now

$$T_{\text{eff}}^{-1}=-(k/4J)\ln\sum_i p_i e^{-4J/kT_i} \tag{2.8}$$

instead of (2.5), while the special choice(10)

$$w^{(i)}=1-\gamma_i''s(x)[s(x+1)+s(x-1)]+(\gamma_i'')^2\,s(x+1)\,s(x-1) \tag{2.9}$$

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[2] We acknowledge a referee for this remark.

with $\gamma_i'' \equiv \tanh(J/kT_i)$, leads to

$$m(1-2\overline{\gamma''})+\overline{(\gamma'')^2}\langle s(x+1)\,s(x)\,s(x-1)\rangle=0 \tag{2.10}$$

where

$$\overline{\gamma''}\equiv\sum_i p_i\gamma_i'', \qquad \overline{(\gamma'')^2}\equiv\sum_i p_i(\gamma_i'')^2 \tag{2.11}$$

which reveals itself as a more difficult case than (2.1) and (2.6) [in particular, (2.10) shows the possibility of having nonzero magnetization states in one dimension for a combination of dynamical mechanisms (2.9) when the distribution of the values of $s(x)$ is non-Gaussian].

3. A MEAN FIELD SOLUTION

Let us consider now the mean field version of the model with transition probabilities (2.1) as defined, for instance, in the Bragg–Williams approximation(6); namely

$$w^{(i)}=\tfrac{1}{2}\alpha[1-\bar{\gamma}_i s(x)] \tag{3.1}$$

where

$$\bar{\gamma}_i\equiv\tanh(Jm/kT_i), \qquad J>0 \tag{3.2}$$

and

$$m=\langle s(x)\rangle_{\text{st}}=N^{-1}[s(1)+s(2)+\cdots+s(N)]_{\text{st}}$$

with the latter equality becoming an identity as $N\to\infty$. It then follows from Eq. (1.3) that

$$dm/d\tau=p(\bar{\gamma}_1-m)+(1-p)(\bar{\gamma}_2-m) \tag{3.3}$$

where $\tau\equiv t\alpha$. This predicts ferromagnetic ordering in the stationary state below a critical (mean) temperature, i.e., when $T\leqslant T_c$, where T_c will be determined below. Before looking explicitly at this fact, however, we notice that the nonlinear equation (3.3) may be expanded around $m=0$ to write

$$dm/d\tau\simeq-(1-A)\,m-Bm^3 \tag{3.4}$$

with

$$A\equiv pK_1^{-1}+(1-p)\,K_2^{-1}, \qquad B\equiv\tfrac{1}{3}[pK_1^{-3}+(1-p)\,K_2^{-3}] \tag{3.5}$$

where we have introduced the notation $K \equiv kT/J$, etc. The corresponding solutions are

$$m = \tilde{A}^{1/2}[\tilde{A}m_0^{-2} + B)\exp(2\tilde{A}\tau) - B]^{-1/2} \tag{3.6}$$

where $m_0 = m(t=0)$, when $\tilde{A} \equiv 1 - A \neq 0$, and

$$m = (2B\tau + m_0^{-2})^{-1/2} \tag{3.7}$$

when $A = 1$. The interpretation of Eqs. (3.6) and (3.7) is similar to that in the equilibrium case: the relaxation from any initial magnetization m_0 slows down as one approaches the condition $A = 1$. This corresponds indeed to the critical temperature K_c, given that the stationary regime for $K < K_c$ $(m \neq 0)$ is characterized, according to Eq. (3.4), by the condition $Bm^2 + (1 - A) = 0$ leading, in particular, to $A \to 1$ as $m \to 0$.

The critical temperature is thus determined by the condition

$$K_c^2 - \Delta K^2 = p(K_c + \Delta K) + (1-p)(K_c - \Delta K) \tag{3.8}$$

or, dropping the alternative minus sign, which happens to be physically irrelevant,

$$K_c = \tfrac{1}{2}\{1 + [1 + 4(2p-1)\,\Delta K + 4\Delta K^2]^{1/2}\} \tag{3.9}$$

This exists for all ΔK and p and has the correct behavior when $p \to 0$, 1, i.e., for the "pure" equilibrium mean field cases at temperatures $K \mp \Delta K$ with critical temperatures $1 \mp \Delta K$, respectively. In particular, one has, for small ΔK,

$$K_c = 1 + (2p-1)\,\Delta K + 4p(1-p)\,\Delta K^2 + \mathcal{O}(\Delta K^3) \tag{3.10}$$

predicting, for instance, very small corrections to the equilibrium case $K_c = 1$ when $p = \frac{1}{2}$. One also has from Eq. (3.3) in the stationary state

$$m = p\tanh(m\,|\,K_1) + (1-p)\tanh(m/K_2) \tag{3.11}$$

or, for small values of m,

$$m \sim \tilde{B}(-\varepsilon)^{\beta}, \qquad \varepsilon \equiv T/T_c - 1 \tag{3.12}$$

with

$$\tilde{B}^2 = B^{-1}[pK_1^{-2} + (1-p)\,K_2^{-2}] \tag{3.13}$$

and $\beta = \frac{1}{2}$.

The above model with $\Delta K = \text{const}$ is thus characterized by a classical critical behavior. Nevertheless, a more involved situation follows (within the present mean field approximation) when ΔK is temperature-dependent. Let $0 \leqslant \Delta K = f(K) \leqslant K$ and, near the critical temperature, $\Delta K = f(K_c) + a\varepsilon + \mathcal{O}(\varepsilon^2)$, where $\varepsilon \equiv K/K_c - 1$ and $a \equiv [f(k)/K]_{K_c} K_c$. Then $K_{1,2} \approx K_{1,2}^c + (K_c \mp a) \in\, > 0$, with $K_{1,2}^c \equiv K_c \mp f(K_c)$. On the other hand, the stationary state near K_c may be characterized, according to Eq. (3.4), as $(1 - A) + Bm^2 - \mathcal{O}(m^4) = 0$, where A and B are the functions defined in Eq. (3.5), which may be expanded as

$$A = 1 + (-\varepsilon)\, a_1 + \varepsilon^2 a_2 + \mathcal{O}(\varepsilon^3) \tag{3.14}$$

with

$$a_1 \equiv p(K_c - a)(K_1^c)^{-2} + (1 - p)(K_c + a)(K_2^c)^{-2} \tag{3.15}$$

$$a_2 \equiv p(K_c - a)(K_1^c)^{-3} + (1 - p)(K_c + a)(K_2^c)^{-3} \tag{3.16}$$

$a_2 > 0$ for all a and p, and $B = B_c + \mathcal{O}(\varepsilon)$, $B_c > 0$, for all a and p. The stationary condition may be written accordingly:

$$m^2 B_c = -a_1 \varepsilon + a_2(-\varepsilon)^2 + \mathcal{O}(m^4, \varepsilon^3, m^2\varepsilon); \qquad a_2, B_c > 0 \tag{3.17}$$

Three cases seem of interest. When $a_1 = 0$, one has from Eq. (3.17) that $m \sim (-\varepsilon)(a_2/B_c)^{1/2}$, implying $\beta = 1$. The case $a_1 > 0$, on the contrary, is again classical, given that it implies $m \sim (-\varepsilon)^{1/2}(a_1/B_c)^{1/2} + \mathcal{O}(\varepsilon)$; the relevant example $K_{1,2} = (1 \mp \delta) K$, $0 < \delta < 1$, so that $\Delta K = f(K) = \delta \cdot K$, belongs to this class, with $a = \delta \cdot K_c$. Finally, when $a_1 < 0$ the solution m from Eq. (3.17) is no longer real. The example $\Delta K \sim \delta \cdot \varepsilon$, $\delta = \text{const}$, near T_c is compatible with the above three cases. One has immediately $K_1^c = K_2^c = K^c = 1$ and $a_1 = 1 + \delta(1 - 2p)$, so that $a_1 = 1$ for $p = \frac{1}{2}$; when $p > \frac{1}{2}$ ($p < \frac{1}{2}$ is symmetrical with $\delta < 0$), a_1 is positive, zero, or negative depending on whether δ is smaller, equal to, or larger than $(2p - 1)^{-1}$.

Let us consider now two interesting limits in the equations above, namely (1) the lowest temperature is zero, $K_1 = 0$, and K_2 is finite, and (2) $K_1 \equiv K$ is finite and $K_2 = K + \Delta K$ is a very high temperature, say $\Delta K = \infty$.

The mean field solution (3.11) reduces in case 1 to

$$m = p + (1 - p) \tanh(m/K_2) \tag{3.18}$$

implying $m > 0$ for all p (>0) and K_2; i.e., a finite temperature K_2 cannot destroy completely the freezing produced by the lowest temperature. In particular, when K_2 is large enough one has from Eq. (3.19) that

$$m \simeq p[1 + (1 - p)/\Delta K] \tag{3.19}$$

Case 2, on the other hand, is characterized by

$$m = p \tanh(m/K) \tag{3.20}$$

becoming nonzero for $K < K_c = p$, i.e., there is always (in the present mean field approximation) a phase transition to states with some spontaneous magnetization in spite of the strong perturbation introduced by the mechanism at infinite temperature.

4. CLUSTER STATISTICS; $d = 1$ AND 2

The basic model in Section 1 may also be solved by following a method used recently by Dickman(4) in a different nonequilibrium problem.(3) The method, which in some sense reduces to the mean-field Bethe approximation for equilibrium problems, rests upon specific rate equations [approximations belonging to the family of the master equation (1.2)] for the density of spins up, $n \equiv N_+/N$, and for the density of up–up pairs of spins, $u = N_{++}/\frac{1}{2}qN$, where q represents the lattice coordination number, with the general structure

$$\frac{da}{dt} = F(a) \equiv \sum_{\text{local conf.}} w(\Delta H)\, P(\text{local conf.})\, \Delta a$$

where a represents n or u, and Δa is the corresponding change in the transition. Such an equation can easily be written explicitly by a detailed counting of (small) cluster configurations, their respective probabilities of occurrence P(local conf.), and the corresponding transition probabilities for the inversion of the central spin.(4)

In order to illustrate the method, we first consider a one-dimensional chain of spins with nearest-neighbor interactions in contact with a single bath at reduced temperature K. There are eight different configurations in this case: $+++$, $-++$, $++-$, $-+-$, $+-+$, $--+$, $+--$, and $---$ with respective probabilities $P = u^2/n$, $u(n-u)/n$, $u(n-u)/n$, $(n-u)^2/n$, $(n-u)^2/(1-n)$, $(n-u)(1-2n+u)/(1-n)$, $(n-u)(1-2n+u)/(1-n)$, and $(1-2n+u)^2/(1-n)$. When the inversion of the central spin is governed by the Metropolis dynamics, i.e., when $w = \min[1, \exp(-\Delta H/kT)]$, the corresponding transition probabilities are $\eta \equiv \exp(-4/K)$ for $+++$ and $---$, and 1 for the rest of the configurations. Thus, one may simply write

$$dn/dt = F_1(n, u; \eta) \tag{4.1}$$

with

$$F_1(n, u; \eta) = \eta\left(\frac{z^2}{v} - \frac{u^2}{n}\right) + 2(n-u)\left(\frac{z}{v} - \frac{u}{n}\right) + (n-u)^2\left(\frac{1}{v} - \frac{1}{n}\right) \tag{4.2}$$

where $v \equiv 1-n$ and $z \equiv 1-2n+u$, and

$$du/dt = F_2(n, u; \eta) \tag{4.3}$$

with

$$F_2(n, u; \eta) = -2\eta\frac{u^2}{n} - 2u\frac{n-u}{n} + 2\frac{(n-u)^2}{v} + 2(n-u)\frac{z}{v} \tag{4.4}$$

To go beyond this pair or first-order mean field approximation, one would write equations similar to (4.1) or (4.3) for the concentration of triplets of spins and introduce the corresponding variables into F_1, F_2, F_3, etc.

Every equilibrium state of the system is characterized in the present approximation by a pair of values for the independent variables n and u satisfying the stationary condition $F_1 = 0$ and $F_2 = 0$ and the corresponding stability condition; it follows that the latter reduces in practice here to the condition

$$(\partial F_1/\partial n)_{n,u} < 0 \tag{4.5}$$

Above the critical temperature T_c, the system may also be characterized by the property that $m = 0$ and, consequently, that $n = v = \frac{1}{2}$ and $u = z$, so that it follows from $F_a = 0$ that

$$u = \frac{1}{2}\frac{1}{1+\eta^{1/2}}, \qquad T \geqslant T_c \tag{4.6}$$

Consequently, the critical point (where the stability of the solution $n = \frac{1}{2}$ breaks down) satisfies

$$(\partial F_1/\partial n)_{n=1/2, u=u_c} = 0 \tag{4.7}$$

or, explicitly,

$$n^{-2}[\eta(1-u)\,u + (\tfrac{1}{2} - u)^2] = 0 \tag{4.8}$$

whose only solution [which is also a solution of Eq. (4.6)] is

$$u_c = \tfrac{1}{2}, \qquad \eta_c = 0 \tag{4.9}$$

That is, the critical temperature for the Ising chain follows here as $T_c = 0$, as in the case of the Bethe approximation when the lattice coordination number is $q = 2$.

The relevance of the above equilibrium example for our purposes rests upon the fact that Eqs. (4.1)–(4.9) hold also in the case of a chain of spins with two competing temperatures; the only difference is that now one has

$$\eta = p \exp[-4/(K - \Delta K)] + (1-p) \exp[-4/(K + \Delta K)] \tag{4.10}$$

Thus, unlike the equilibrium case, η is positive for any p such that $0 < p < 1$ and $\Delta K \neq 0$ and, as a consequence, there is no critical point at all, i.e., Eq. (4.5) is always fulfilled.

This compels one to consider the two-dimensional version of the model in the same approximation. The relevant functions in this case are

$$F_1 = (n-u)^4 \left(\frac{1}{v^3} - \frac{1}{n^3} \right) + 4(n-u)^3 \left(\frac{z}{v^3} - \frac{u}{n^3} \right) + 6(n-n)^2 \left(\frac{z^2}{v^3} - \frac{u^2}{n^3} \right) \\ + 4\eta_1 (n-u) \left(\frac{z^3}{v^3} - \frac{u^3}{n^3} \right) + \eta_2 \left(\frac{z^4}{v^3} - \frac{u^4}{n^3} \right) \tag{4.11}$$

and

$$\frac{1}{4} F_2 = \frac{(n-u)^4}{v^3} + (n-u)^3 \left(\frac{3z}{v^3} - \frac{u}{n^3} \right) + 3(n-u)^2 \left(\frac{z^2}{v^3} - \frac{u^2}{n^3} \right) \\ + \eta_1 (n-u) \left(\frac{z^3}{v^3} - \frac{3u^3}{n^3} \right) - \eta_2 \frac{u^4}{n^3} \tag{4.12}$$

where $\eta_1 = p\eta + (1-p)\eta'$ and $\eta_2 = p\eta^2 + (1-p)\eta'^2$, with $\eta = \exp[-4/(K - \Delta K)]$ and $\eta' = \exp[-4/(K + \Delta K)]$ for Metropolis dynamics. For $T \geqslant T_c$ (i.e., $m = 0$: $n = v = \frac{1}{2}$ and $u = z$), one has in the stationary state

$$(1-2u)^3 (1+2u) - 16u^3[u(\eta_2 - 2\eta_1) + \eta_1] = 0 \tag{4.13}$$

giving $u(T \geqslant T_c)$ as a function of K, ΔK, and p, while the condition (4.7) produces

$$\eta_2^c u_c^3 (\tfrac{3}{2} u_c - 1) - 6\eta_1^c u_c^2 (\tfrac{1}{2} - u_c)^2 + 3u_c (\tfrac{1}{2} - u_c)^2 (3u_c - 1) \\ + (\tfrac{1}{2} - u_c)^3 (6u_c - 1) + \tfrac{3}{2} (\tfrac{1}{2} - u_c)^4 = 0 \tag{4.14}$$

When $u = u_c = 1/3$, Eqs. (4.13) and (4.14) reduce to a single equation,

$$5 - 16(\eta_1^c + \eta_2^c) = 0 \tag{4.15}$$

This characteries the critical temperature as a function of p and ΔK. Figures 1 and 2 represent the variation of K_1^c with p and ΔK. Figure 3, on the other hand, represents the coexistence curve $m = m(K_1)$ for different values of ΔK and p obtained as a solution of $F_1 = 0$ and $F_2 = 0$.

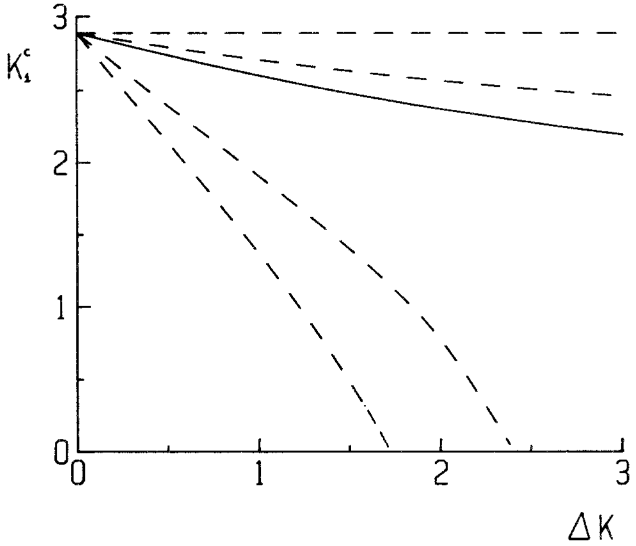

Fig. 1. The critical temperature K_1^c as a function of ΔK, from Eq. (4.15), for different values of p: $p = 1$, 0.9, 0.84375, 0.5, and 0.25 from top to bottom.

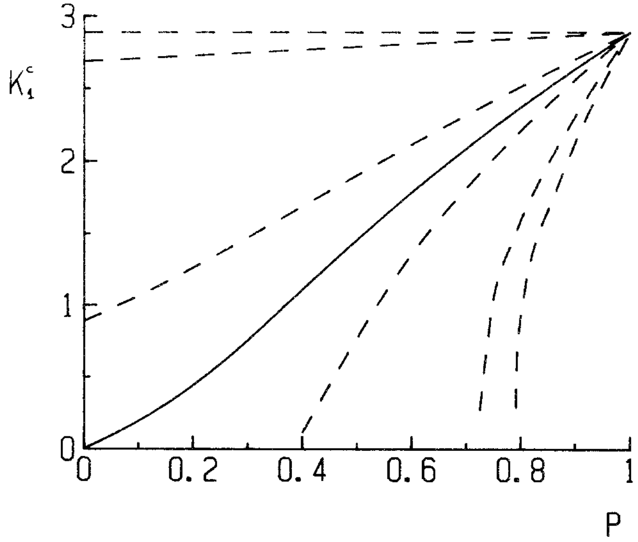

Fig. 2. Plot of K_1^c as a function of p, from Eq. (4.15), for different values of ΔK: $\Delta K = 0$, 0.1, 1, 1.442695, 2, 5, and 10 from top to bottom.

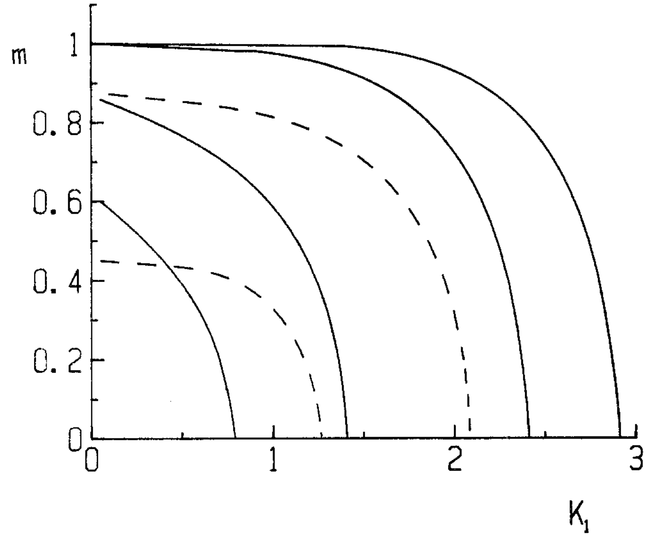

Fig. 3. The coexistence curve $m = m(K_1)$ for the two-dimensional model with two competing temperatures for different values of ΔK and p. The curves are, from top to bottom, as follows: (—) $p = \frac{1}{2}$ and $\Delta K = 0$, 0.5, 1.5, and 2; (- -) $p = 0.8$ and $\Delta K = 2.5$ and 3.75.

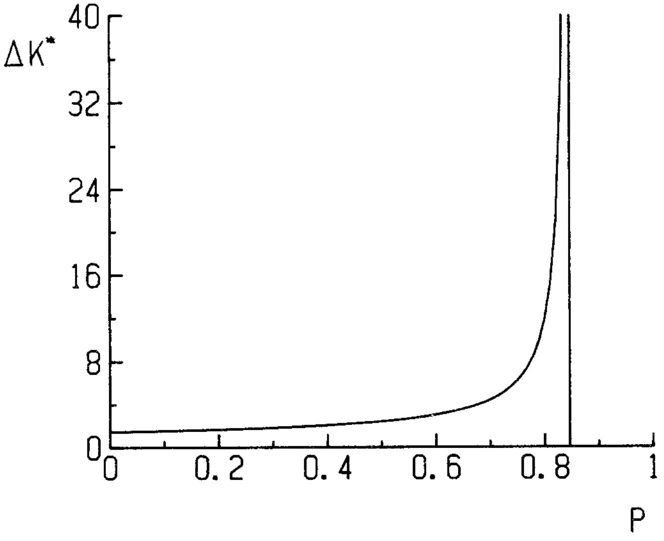

Fig. 4. The quantity ΔK^* producing a zero critical value for the lowest temperature, $K_1^c = 0$, as a function of p. The vertical line is the asymptote $p^* = 27/32$. Thus, only the region below and to the right of the solid line $\Delta K^*(p)$ corresponds to the existence of a positive critical temperature K_1^c.

Those figures reveal an interesting fact. Figure 1, for instance, shows a monotonic decrease of K_1^c toward zero with increasing ΔK when p is smaller than, say, p^* (represented by a solid line in the figure), revealing that there is no positive solution $K_1^c(p, \Delta K)$ when $\Delta K > \Delta K^*(p)$. The function $K^*(p)$ is depicted in Fig. 4. The above value for p^* can be obtained by making $K_1^c = 0$ in Eq. (4.15); it follows that the values of K_2^c for which $K_1^c = 0$ are given by

$$\eta_c^1 \equiv \exp(-4/K_2^c) = -\tfrac{1}{2} + \tfrac{1}{2}[1 + 5/4(1-p)]^{1/2} \tag{4.16}$$

which cannot be satisfied ($\eta_c^1 \leqslant 1$) when $p \geqslant p^* = 27/32$.

5. APPROXIMATE CORRELATIONS

Let us consider in this section the two-dimensional version of the model with nearest neighbor interactions characterized by a transition probability given by

$$\begin{aligned} w^{(i)} = \frac{\alpha}{2}\left[1 - \frac{1}{2}\gamma_i s_{x,y}(s_{x+1,y} + s_{x-1,y})\right] \\ \times \left[1 - \frac{1}{2}\gamma_i s_{x,y}(s_{x,y+1} + s_{x,y-1})\right] \end{aligned} \tag{5.1}$$

where $i = 1, 2$; γ_i is defined in Eq. (2.2); $T_{1,2} = T \mp \Delta T$; and x (y) represents the index describing the lattice sites in the horizontal (vertical) direction. The properties of the stationary state follow from Eq. (1.7) as

$$\begin{aligned} m[1 + 2p(\gamma_2 - \gamma_1) - 2\gamma_2] + \tfrac{1}{4}[p\gamma_1^2 + (1-p)\gamma_2^2] \\ \langle s_{x,y}(s_{x+1,y} + s_{x-1,y})(s_{x,y-1} + s_{x,y+1})\rangle_{\mathrm{st}} = 0 \end{aligned} \tag{5.2}$$

In principle, the correlation functions in this expression may be treated approximately by assuming that their nonequilibrium behavior at temperature T is the weighted arithmetic mean of the corresponding exact equilibrium behaviors at temperatures T_1 and T_2 (i.e., with weights p and $1-p$, respectively), with these obtained from the Onsager solution. This amounts in particular to the neglect of some nonequilibrium effects in the correlation functions, but, as the approximation becomes exact in the limit $\Delta T \to 0$, one may still hope to get a good description for small values of ΔT; of course, the most important deviations from the real behavior will show up near the critical temperature, where the situation is essentially governed by the correlations.

Let $m_0(T)$ represent the Onsager solution at temperature T,

$$m_0(T) = [1 - (1 - \gamma^2)^2 \gamma^{-4}]^{1/8} \tag{5.3}$$

One has from Eq. (1.3) for the equilibrium state

$$m_0(T_i)(1 - 2\gamma_i) + \tfrac{1}{4}\gamma_i^2 \langle s(s+s)(s+s) \rangle_i = 0 \tag{5.4}$$

where $s(s+s)(s+s)$ is a simplified notation for the averaged quantity in Eq. (5.2) and $\langle \cdots \rangle_i$ represents the equilibrium average at temperature T_i. Our assumption is thus

$$\begin{aligned} &\langle s(s+s)(s+s) \rangle_{\mathrm{st}} \\ &\quad = -4p\gamma_1^{-2} m_0(T_1)(1 - 2\gamma_1) - 4(1-p)\,\gamma_2^{-2} m_0(T_2)(1 - 2\gamma_2) \end{aligned} \tag{5.5}$$

which can be used in Eq. (5.2) to obtain the basic equation for the model:

$$\begin{aligned} m(T) = \frac{p\gamma_1^2 + (1-p)\,\gamma_2^2}{1 + 2p(\gamma_2 - \gamma_1) - 2\gamma_2} &[p\gamma_1^{-2} m_0(T_1)(1 - 2\gamma_1) \\ &+ (1-p)\,\gamma_2^{-2} m_0(T_2)(1 - 2\gamma_2)] \end{aligned} \tag{5.6}$$

This may easily be generalized along two different lines: the consideration of more than two temperatures, as indicated in Section 1, and the introduction of different coupling constants, say J_x and J_y, for the two principal directions of the lattice.

The expansion of Eq. (5.6) for small values of ΔT produces after some algebra the result

$$m(T) = m_0(T) - (2p - 1)\,\Delta T\,\partial m_0/\partial T + \cdots \tag{5.7}$$

where the term of order ΔT^{a}, which represents the leading deviation from the equilibrium value m_0 when $p = \frac{1}{2}$, is a complicated function of p, γ, m_0, $\partial m_0/\partial T$, and $\partial^2_{m_0}/\partial T^2$; we avoid writing it explicitly here because it turns out to be more convenient to use Eq. (5.6) directly for practical purposes, and also because the range of validity of the present approximation reduces in practice to very small values of ΔT. The interest of Eq. (5.7), which has a singularity at $T = T_c^{\mathrm{onsager}}$, where $\partial m_0/\partial T$ is not well defined, rests upon the fact that one can see explicitly how the consideration of some nonequilibrium effects introduces corrections to the equilibrium behavior $m_0(T)$. It also allows one to find a criterion for the validity of this approximation; that is, from the condition $(2p - 1)\,\Delta T(\partial m_0/\partial T) \ll m_0(T)$ one has the requirement

$$T \ll T_c^{\mathrm{onsager}} - \tfrac{1}{8}(2p - 1)\,\Delta T \tag{5.8}$$

implying, in particular, that one should not trust the critical behavior (e.g., values for the critical indexes) following from this approximation. The overall behavior predicted by Eq. (5.6) only compares well with the Monte Carlo data for $\Delta T \ll 0.05$. A better approximation would require the introduction of the concept of an effective temperature (along the lines of the argument at the end of Section 2) together with an appropriately renormalized Hamiltonian, a task that goes beyond the scope of the present paper.

6. A MONTE CARLO EXPERIMENT

The results obtained via different approximations in the preceding sections can be compared with the behavior of the basic model in a series of Monte Carlo experiments. The model in Section 1 was implemented in the computer by choosing at random with probability P_i the temperature T_i that enters into the transition probability for the inversion of the spin at the randomly selected lattice site. We shall refer to the case $T_1 = T - \Delta T$, $T_2 = T + \Delta T$, $p_1 = p_2 = \frac{1}{2}$, and Metropolis dynamics. The system is a square lattice of size 96×96 (and sometimes 152×152). We used the so-called multispin coding(11,12) and a shift-register random-number generator,(13) namely $Y_i = \text{XOR}\ (Y_{i-q+p}, Y_{i-q})$ with $p = 103$, $q = 250$; the former was written in PROFESSIONAL FORTRAN for IBM PCs with 8087 coprocessors. The total data required 3480 hr in our student computer lab.

The magnetization data for 96×96 lattices are presented in Fig. 5. Our first observation here refers to the finite-size effects, which are apparently larger than in the equilibrium (i.e., $\Delta T = 0$) case, e.g., the finite-size tails in Fig. 5 are similar to the ones when the lattice is 60×60 for $\Delta T = 0$, and they decrease by 15% when we used instead 152×152 lattices. We expect to study those finite-size effects in the near future in more detail, and concentrate now mostly on the qualitative aspects of the phenomena. The situation depicted by Fig. 5 is, on the other hand, rather clear: $\Delta T = 0.05$ produces unobservable differences from the equilibrium Onsager case, while $\Delta T = 0.5$ shows a definite shift of the critical temperature toward $T_c(\Delta T = 0.5) = 2.25 \pm 0.01$ J/k. As expected, the mean field approximation in Section 4 only describes the data qualitatively (cf. Fig. 5); note, however, that some of the differences between mean field and Monte Carlo results are probably just a consequence of the differences in dynamics implied by the mean field approximation (cf. last paragraph in Section 2). The approximation in Section 5 is more accurate, but it fails to reproduce the data near T_c even for $\Delta T = 0.05$.

The energy data are represented in Fig. 6; this basically confirms our observations above. In particular, Fig. 6 reveals that the only differences

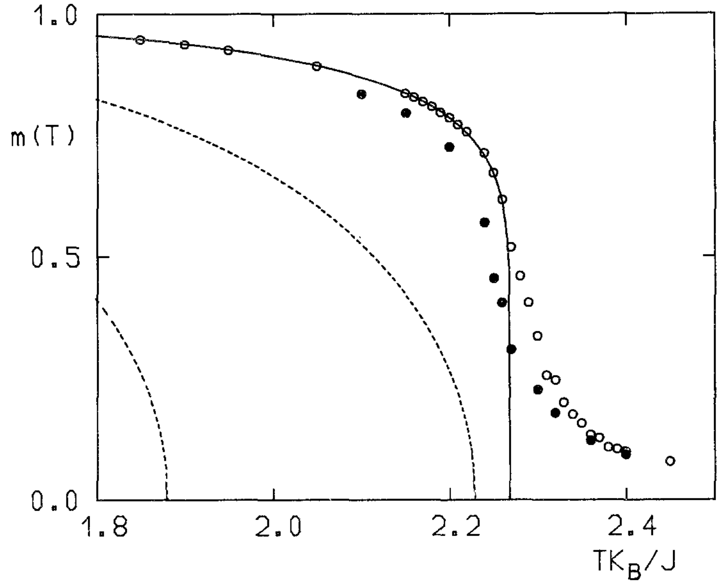

Fig. 5. The system magnetization as a function of the mean temperature in the case of a series of Monte Carlo experiments for $\Delta T=$ (○) 0.05 and (●) 0.5 $p_1=p_2=\frac{1}{2}$, compared with (—) the equilibrium result and (- -) the respective predictions in Section 4. The dashed lines are also for $\Delta T=$ (upper) 0.05 and (lower) 0.5, and they are normalized horizontally to the Onsager critical temperature.

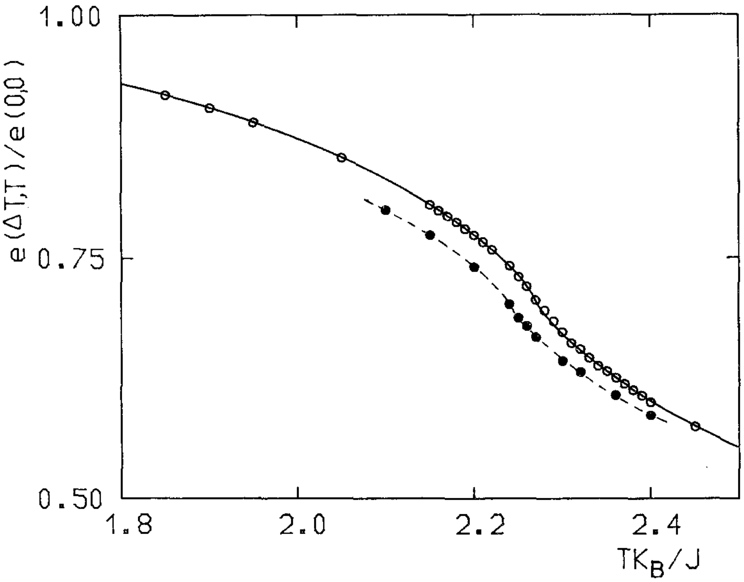

Fig. 6. The system energy as a function of temperature in the case of the Monte Carlo experiments; same symbols as in Fig. 5. (—) The Onsager (equilibrium) result; (- -) a cubic interpolation to the data.

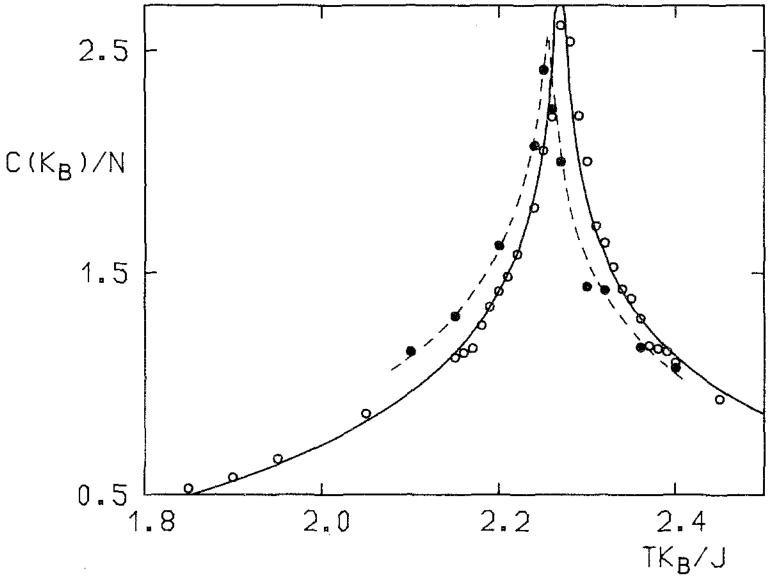

Fig. 7. The specific heat as a function of temperature; (○, ●; same notation as in Fig. 5) the values obtained from the energy fluctuations; (—; - -) results from a temperature derivative of the energy data in Fig. 6 (after performing cubic interpolations of those data).

between the data for $\Delta T = 0.05$ and the Onsager result occur extremely near the critical temperature. Figure 7 shows the behavior of the specific heat obtained from the fluctuations of the energy; this is also compared in the figure with the behavior of the temperature derivative of the energy; both computations give the same result for the present nonequilibrium problem.

ACKNOWLEDGMENTS

We acknowledge useful discussions with Joel L. Lebowitz and the suggestions from the referees, which prompted us to make more explicit some results in Sections 2 and 4. This work was partially supported by the CAICYT, Project PB85-0062 and by the U.S.–Spanish Cooperative Research Program, grant CCB-8402/025.

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